Meandric systems and the Infinite Noodle Soup

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Lecture notes
XIth Mexican Probability Summer School

Mexico City

July 2026

Table of contents

Chapter 1

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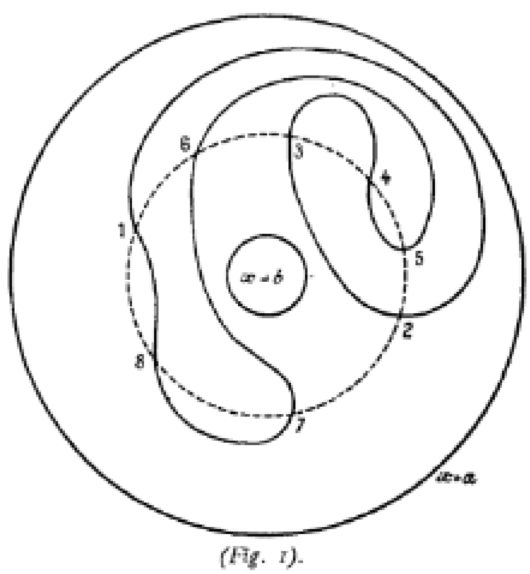


Figure 1.1: Figure taken from H. Poincaré's paper [14] (1912).

A meandric system is a configuration of loops in the plane which do not cross, and which intersect the horizontal axis in a given fixed number of points.
The most interesting class of meandric systems is most likely *meanders*, which are meandric systems with only one loop. The first mention of such objects dates back to Henri Poincaré in his last mathematical article, at the beginning of the 20th century. In that paper, Poincaré asked the following seemingly simple question: in how many different ways can a simple closed curve (called meander, see the thick curve on Fig. 1.1) cross a line (the dotted line on Fig. 1.1) a specified number of times? Despite the simplicity of the question, there is no known formula. During the 20th century, mathematicians discovered connections between meanders and statistical physics; using these connections, physicists conjectured the asymptotic number of meanders

crossing the line at n points, as $n \to \infty$. These predictions seem to match the numerical simulations.
General meandric systems, with possibly several loops, have been attracting more attention since the beginning of the 21st century, both from combinatorists and probabilists. They are indeed combinatorially simpler to study, and have at the same time a very rich structure. Connections were discovered with e.g. noncrossing partitions, free probability theory and random geometry.

The goal of this mini-course is to investigate the behaviour of a large random meandric system. In particular, we will define and study the *Infinite Noodle Soup*, which can be seen as the limit of a large meandric system chosen uniformly at random.

1.1 Definitions

1.1.1 Definition of the model

In order to properly define mandric systems, we start by defining *noncrossing matchings*, which play an important role in what follows.

Definition 1.1.1. *A noncrossing matching of size n is a matching of the integers $\{1, \ldots, 2n\}$ by (non-oriented) arcs above the horizontal axis, so that the arcs do not cross. See Fig. 1.2 for an example of a noncrossing matching of size 5.*

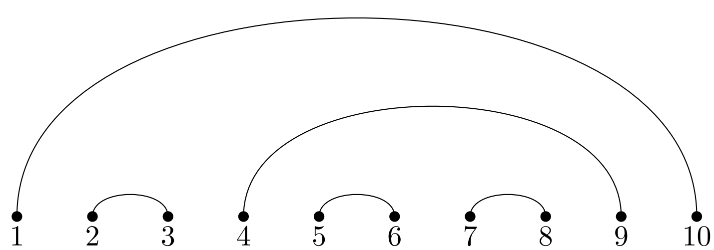


Figure 1.2: A noncrossing matching of size 5.

We can now define meandric systems.

Definition 1.1.2. *A meandric system of size n is made of two noncrossing matchings of size n, one above the axis and the other below the axis.*
Alternatively, it is a configuration of closed loops on the integers $\{1, \ldots, 2n\}$ so that the loops do not cross. See Fig. 1.3 for an example of two noncrossing matchings of size 5, and the associated meandric system.

Remark 1.1.1. *We will also sometimes define noncrossing matchings and meandric systems on a finite interval $I \subseteq \mathbb{Z}$ if even length, with the natural analogue definitions.*

In order to study properties of large random meandric systems, we will develop different tools from combinatorics and probability theory. In the rest of the paper, we denote by m_n a meandric system of size n chosen uniformly at random.

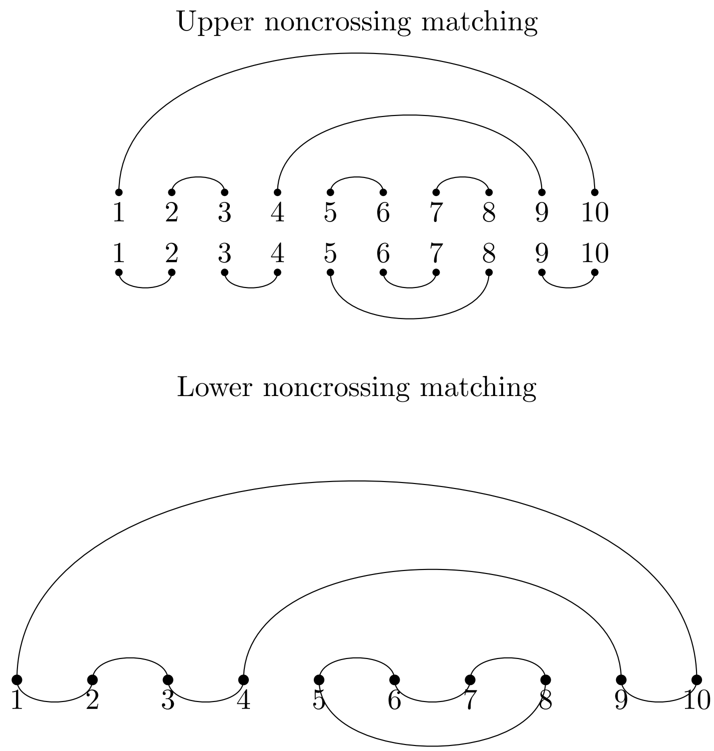


Figure 1.3: Top: two noncrossing matchings. Bottom: the resulting meandric system.

1.1.2 Motivations

Let us start by listing some of the main motivations for the study of meandric systems.

Meanders As said already, the first motivation goes back to Poincaré. He was interested in counting a certain type of meandric systems called meanders, which are meandric systems with only one loop. Counting meanders is a notoriously hard problem, and we refer the interested reader to [16] for a survey of this topic. Fig. 1.4 lists all meanders of size at most 3.

The asymptotic number of meanders of size n was conjectured some years ago:

Conjecture 1.1.3 (Di Francesco, Golinelli, Guitter [6])**.** *The number $\mathcal{A}_n$ of meanders of size n behaves asymptotically like*

$$\mathcal{A}_n \sim C\rho^n n^{-\alpha}$$

for some $C > 0, \rho > 1$ and $\alpha = \frac{29+\sqrt{145}}{12}$.

The exponential growth of $\mathcal{A}_n$, that is, the existence of a constant $\rho > 0$ such that $\mathcal{A}_n^{\frac{1}{n}} \to \rho$ as $n \to \infty$, follows from a supermultiplicativity argument, see Exercise sheet. The striking part of the conjecture is the value of this exponent α controlling the polynomial correction for $\mathcal{A}_n$. This conjecture is supported by physics heuristics and numerical evidence (see e.g. [10]), although no proof has been found yet. Simulations provide $\rho \approx 12.26$.

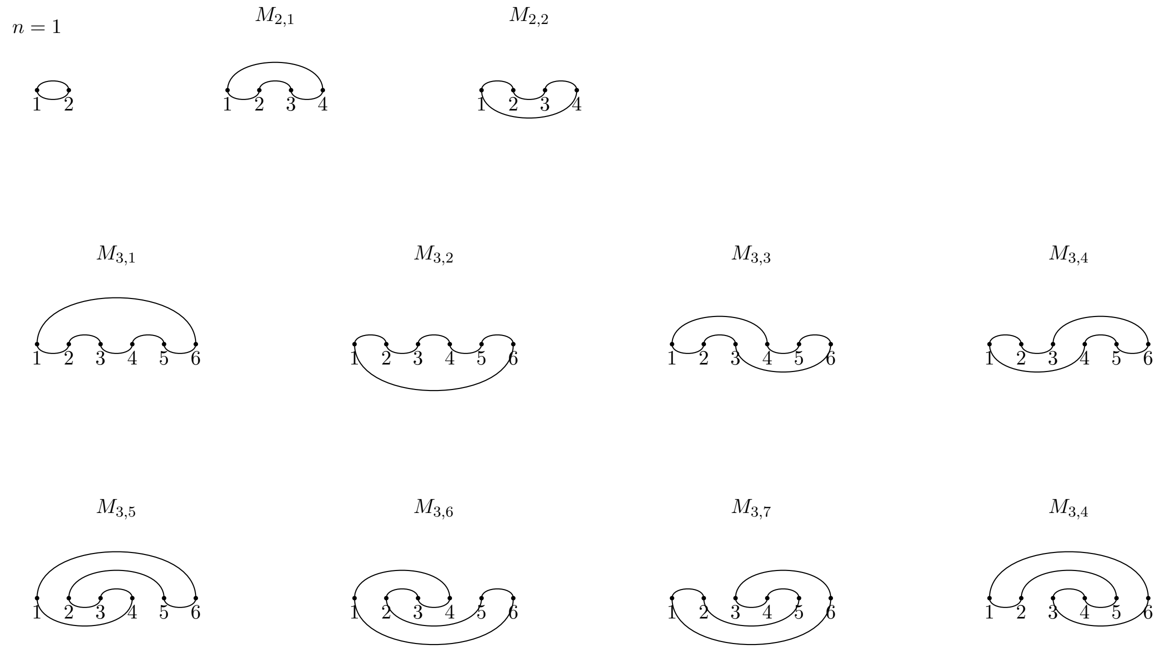


Figure 1.4: All meanders of size $1, 2$ and 3.

Combinatorics

The second motivation is the multiple connections with combinatorics problems. For instance, meandric systems are in bijection with pairs of noncrossing partitions of an integer [9] (see Exercise sheet).

Free probability

Some aspects of free probability theory can be rephrased in terms of meandric systems. More precisely, Nica [13] studied a certain subset $\mathcal{M}_n^{(irr)}$ of meandric systems which he called *irreducible*. He proved that the generating function $\sum_{n\geq 0} |\mathcal{M}_n^{(irr)}| x^{2n}$ can be seen as the R-transform of $\xi\eta$, where ξ, η are centered random variables with the semicircle law. The interested leader can read about it in [13, Section 4].

Random geometry

A meandric system can be naturally seen as a metric space: all arcs have length 1, and i and $i+1$ are connected by an edge of length 1 for all $1 \leq i \leq 2n$. Then, a random uniform meandric system of size n is conjectured to converge, as a (random) metric space, after rescaling of distances, towards a limiting object [2]. This leads to several conjectures, among which the following ([2, Conjecture 1.3]):

Conjecture 1.1.4. *For m a meandric system, let $BL(m)$ be the number of vertices in the largest loop of m. Then, we have as $n \to \infty$:*

$$BL(m_n) = n^{\alpha + o(1)},$$

where m_n is a uniform meandric system of size n and

$$\alpha = \frac{3 - \sqrt{2}}{2} \approx 0.7929.$$

Local limit We will be interested in another sort of limit: the *local* limit of m_n, as $n \to \infty$. We will show that m_n converges in a local sense to a random infinite object called the Infinite Noodle Soup (INS), introduced in [4]. In particular, proving this convergence will allow us to derive several asymptotic structural properties of m_n, such as its typical number of loops.

1.2 Combinatorial study of meandric systems

We will start by computing the number of meandric systems of a given size. A first step is to enumerate noncrossing matchings. For all $n \geq 1$, denote by NC_n the set of noncrossing matchings of size n, and by $\mathcal{M}_n$ the set of meandric systems of size n.

Lemma 1.2.1. *For all $n \geq 1$, $|NC_n| = Cat_n$, where $Cat_n := \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.*
In particular, we have by Stirling's formula that $|NC_n| \sim \frac{1}{\sqrt{\pi}} 4^n n^{-3/2}$ as $n \to \infty$.

As a corollary, we can enumerate meandric systems, since a meandric system is made of two noncrossing matchings.

Corollary 1.2.2. *There are $|\mathcal{M}_n| = Cat_n^2$ meandric systems of size n. In particular, we have that $|\mathcal{M}_n| \sim \frac{1}{\pi} 16^n n^{-3}$ as $n \to \infty$.*

Catalan numbers are a central object in combinatorics. They are known to enumerate many families of objects: proper bracket structures with $2n$ brackets, trees with n vertices, Dyck paths with $2n$ steps, ... The sequence of Catalan numbers is characterised by the following induction equation: it is the only sequence $(u_n)_{n\geq 0}$ that satisfies $u_0 = u_1 = 1$ and, for all $n \geq 1$, $u_{n+1} = \sum_{k=0}^{n} u_k u_{n-k}$.

Proof of Lemma 1.2.1. We prove the result by induction on n.

For $n = 0$ there is one noncrossing matching on the empty set, the empty matching. For $n = 1$, there is only one matching on $\{1, 2\}$, which connects 1 and 2.

Now fix $n \geq 1$. Let $P \in NC_{n+1}$ be a noncrossing matching of size $n + 1$, that is, on $\{1, \dots, 2n + 2\}$, and consider the integer j which is matched with 1 in P. Then, since P is noncrossing, necessarily there must be an even number of integers between 1 and j, so we can write $j = 2k+2$ for some $k \in \{0, \dots, n\}$. Furthermore, the arcs under the arc $(1, j)$ must form a noncrossing matching (on $\{2, \dots, 2k+1\}$, possibly empty if $j = 2$), and the rest must also form a noncrossing matching (on $\{2k + 3, \dots, 2n\}$, possibly empty if $k = n + 1$). Hence, we get:

$$|NC_{n+1}| = \sum_{k=0}^{n} |NC_k| \times |NC_{n-k}|.$$

Thus, $(|NC_n|)_{n\geq 0}$ satisfies: $|NC_0| = |NC_1| = 1$ and $|NC_{n+1}| = \sum\limits_{k=0}^{n} |NC_k| \times |NC_{n-k}|$.

It has the same initial conditions and satisfies the same recurrence relation as the sequence of Catalan numbers, so we have the equality for all $n \geq 0$. □

1.2.1 Uniform meandric systems

Since the set of meandric systems of a given size is finite, we can define a random uniform element in this set.

Definition 1.2.3 (Uniform meandric system)**.** *For all $n \geq 1$, we denote by m_n an element of $\mathcal{M}_n$ chosen uniformly at random, among the Cat_n^2 elements of the set. Equivalently, m_n is obtained by taking two i.i.d. uniform noncrossing matchings, and putting one above the axis and one below.*

Our aim is to understand the asymptotic properties of m_n, as $n \to \infty$. An example of question we want to answer is the following, raised independently by combinatoricians (Goulden, Nica and Puder [9]) and probabilists (Kargin [11]).

Question 1.2.4. *What is the average number of loops in a random uniform meandric system of size n?*

With Valentin Féray [8], we provide an answer to this natural question. Let $\mathcal{L}(m)$ be the number of loops of a meandric system m.

Theorem 1.2.5. *As $n \to \infty$, we have*

$$\mathbb{E}[\mathcal{L}(m_n)] \sim \kappa n,$$

for some constant $\kappa \in (0, 1)$.

Actually we obtain a stronger convergence result:

Theorem 1.2.6. *We have*

$$\frac{\mathcal{L}(m_n)}{n} \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} \kappa.$$

In other words, the number of loops of a uniform meandric system is very concentrated around its expectation. With the help of a computer software, we get the bounds $0.207 \leq \kappa \leq 0.292$, while simulations suggest $\kappa \approx 0.23$. An interesting open problem, which we partially tackle in Section 3.2, is to compute the exact value of the constant κ of Theorems 1.2.5 and 1.2.6.

We will only prove Theorem 1.2.5. The main idea of the proof will be to define the limit of $(m_n)_{n\geq 1}$ in a *local* sense, which should be thought of as an "infinite random meandric system". This object is called the Infinite Noodle Soup (INS), and was introduced by Curien, Kozma, Sidoravicius and Tournier [4]. The constant κ can then be expressed on the INS directly.

1.2.2 Intuition on the theorem

Let us immediately provide an intuition of why the expected number of loops in m_n should indeed grow linearly with n. Consider the smallest possible loop, where two neighbouring vertices are connected above and below the axis (see Fig. 1.5); let $S(m)$ be the number of such small loops in a meandric system m. We have, writing $i \leftrightarrow j$ for "i is connected to j":

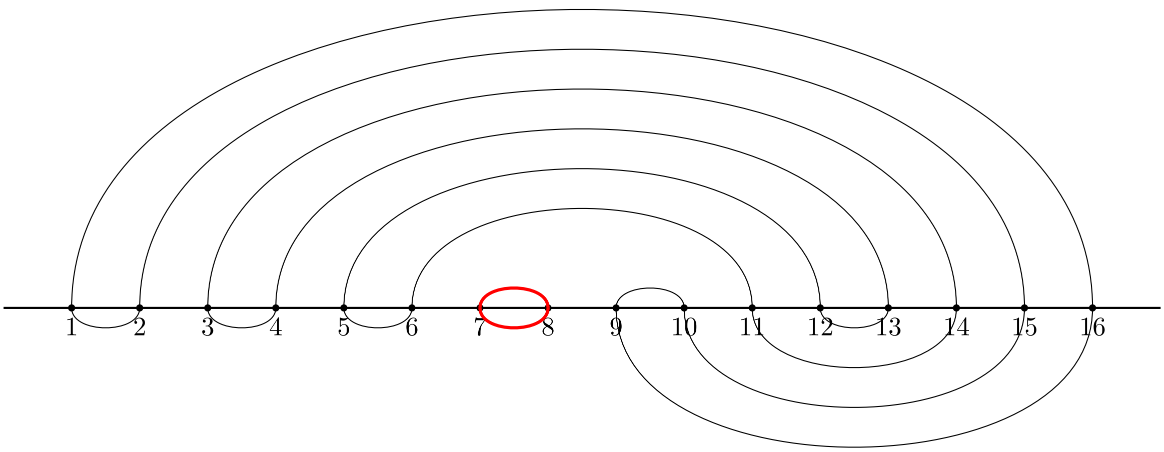


Figure 1.5: A meandric system of size 8, with a small loop highlighted in red.

$$
\begin{aligned}
\mathbb{E}[S(m_n)] &= \mathbb{E}\left[\sum_{i=1}^{2n-1} \mathbb{1}_{i\leftrightarrow i+1 \text{ above and below the axis}}\right] \\
&= \sum_{i=1}^{2n-1} \mathbb{P}(i \leftrightarrow i+1 \text{ above and below the axis }) \\
&= \sum_{i=1}^{2n-1} \frac{(\#\text{ matchings of size } n \text{ with } i \leftrightarrow i+1)^2}{(\#\text{ matchings of size } n)^2} \\
&= (2n-1)\,\frac{Cat_{n-1}^2}{Cat_n^2} \\
&\underset{n\to\infty}{\sim} (2n-1)\frac{16^{n-1}}{\pi(n-1)^3}\frac{\pi n^3}{16^n} \text{ by Stirling's formula} \\
&\underset{n\to\infty}{\sim} 2n\,16^{-1} \\
&= \frac{n}{8}.
\end{aligned}
$$

Here we have used the fact that, for all i, the number of noncrossing matchings of size n with i connected to $i+1$ is just the number of matchings of size $(n-1)$. Indeed, one only needs to connect the points $\{1,\dots,i-1,i+2,\dots,n\}$ in a noncrossing way.

Therefore, there are asymptotically $\frac{n}{8}$ small loops in m_n, on average.

Intuitively, we could do this for all possible shapes, and take the sum.

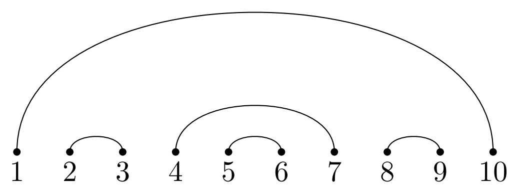
Remark 1.2.1. *With a bit of additional work, this could be turned into a rigorous proof of Theorem 1.5.1. However, we will take another route, characterise the local limit of m_n and show that the constant κ actually corresponds to an observable of the infinite noodle soup.*

1.3 Important tool: bijection with pairs of Dyck paths

A very useful point of view on meandric systems consists in seeing them as pairs of *Dyck paths.*

Definition 1.3.1 (Dyck path)**.** *For $n \geq 1$, a Dyck path of semi-length n is a path starting from $(0,0)$, ending at $(2n,0)$, with steps in $\{(1,1),(1,-1)\}$, and always staying (weakly) above the horizontal axis. We define D_n as the set of Dyck paths of semi-length n. See Fig. 1.6, right, for an example of a Dyck path of smei-length 5.*

Dyck path are important objects in combinatorics. They are also counted by Catalan numbers ($|D_n| = Cat_n$ for all $n \geq 1$). They are used in algebra (representation theory), in the study of random trees (Dyck paths code the contour of trees), or in computer science (language theory).



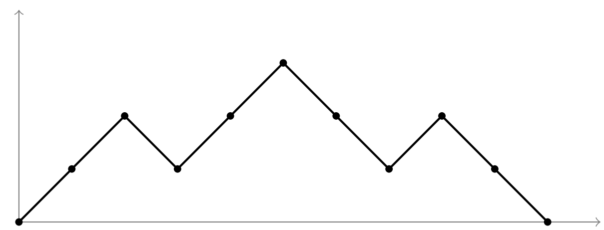


Figure 1.6: Left: a noncrossing matching of size 5. Right: the Dyck path of semi-length 5 associated by the natural bijection.

It turns out that Dyck paths of a given semi-length are in bijection with noncrossing matchings of the same size, see Fig. 1.6. There is actually an explicit, natural canonical bijection, which we will use a lot in what follows.

Proposition 1.3.1. *For all n, the sets NC_n and D_n are in bijection.*

Proof. Consider a noncrossing matching. We construct from it a Dyck path as follows. Read the matching from left to right ; every time you see an integer matched to its right, make an up step $(1,1)$. Every time you see an integer matched to its left, make a down step $(1,-1)$.

Now observe that, for any integer matched to its left that you see, you have already seen the integer matched with it. Hence, we have always made at least as many up steps as down steps. Therefore, the path constructed is always weakly above the axis. Since there are as many integers matched to their left as to their right, the ending point is indeed $(2n,0)$. Finally, one can recover the meandric system from its image by turning steps $(1,1)$ (resp. $(1,-1)$) into half-arcs going to the right (resp. to the left), and then connect the half-arcs in the unique possible noncrossing way. Thus, the construction is indeed a bijection. □

As a consequence, meandric systems of size n are in bijection with pairs of Dyck paths of semi-length n.

We will also need for later purposes to enumerate prefixes of Dyck paths, that is, paths that start from $(0,0)$, end at some $(i,j) \in \mathbb{N}^2$, and always stay weakly above the horizontal axis. See Fig. 1.7 for an example of a Dyck prefix from $(0,0)$ to $(8,2)$.

Define now, for any $a, b \in \mathbb{N}^2$, $S_{a \to b}$ as the number of paths with steps $\{(1,1),(1,-1)\}$ that stay (weakly) above the horizontal axis.

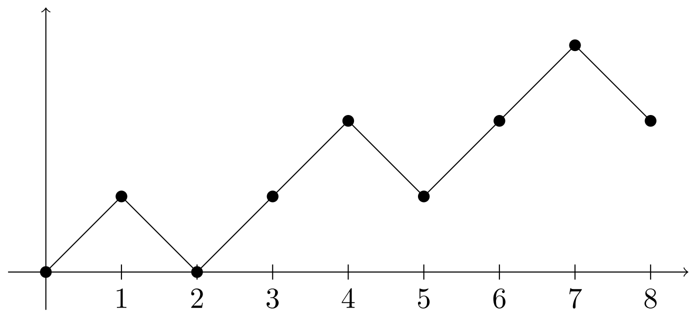


Figure 1.7: A Dyck prefix from $(0,0)$ to $(8,2)$.

Proposition 1.3.2. *Let $i \geq j \geq 0$ be such that $i+j = 0 \mod 2$. Then, we have:*

$$S_{(0,0)\to(i,j)} = \frac{j+1}{i+1}\binom{i+1}{\frac{i-j}{2}}.$$

Proof. The proof uses the reflection principle. See Exercise sheet. □

1.3.1 A third point of view

We also consider a third way to see noncrossing matchings, as a sequence of elements of $\{+1,-1\}$. For any $P \in NC_n$, any $x \in \{1,\dots,2n\}$, set $\omega_x = 1$ if x is connected to its right, and $\omega_x = -1$ if it is connected to its left. In other words, ω_x is the x-th step in the Dyck path associated to P. For example, the Dyck path in Fig. 1.6 is associated to the sequence $(+1,+1,-1,+1,+1,-1,-1,+1,-1,-1)$.

Proposition 1.3.3. *The set NC_n is in bijection with the set of sequences of $\{+1,-1\}$ of length $2n$, with n '$+1$' and n '-1', and so that , for all $1 \leq k \leq 2n$, there are at least as many '$+1$' as '-1' in the first k elements of the sequence.*

The proof is direct from Proposition 1.3.1, which also provides a canonical bijection between the two sets.

1.4 Convergence to the Infinite Noodle Soup

We now introduce the so-called Infinite Noodle Soup (INS), which is the limit of a uniform meandric system of size n as $n \to \infty$ in a local sense that we will define in 1.4.4. Intuitively, the INS should be thought of as a meandric system defined on the whole axis $\mathbb{Z}$.

As in the finite case, we start by defining *random infinite noncrossing matchings.* As for finite noncrossing matchings, there are again three different ways to look at infinite matchings; perhaps the most elegant is as a bi-infinite sequence of $\{+1,-1\}$.

Definition 1.4.1 (Random infinite noncrossing matching)**.** *Consider a random bi-infinite sequence $(\omega_)i_{i\in\mathbb{Z}}$ of i.i.d. variables taking the value $+1$ with probability $1/2$, and the value -1 with probability $1/2$ (that is, they follow a Rademacher distribution).*

For all $i \in \mathbb{Z}$, if $\omega_i = 1$, start an arrow above the horizontal axis pointing to the right. Otherwise, start an arrow pointing to the left. Then, connect the integers of $\mathbb{Z}$ above the axis, in the only possible noncrossing way. See Fig. 1.8 for an example on a small slice of $\mathbb{Z}$.

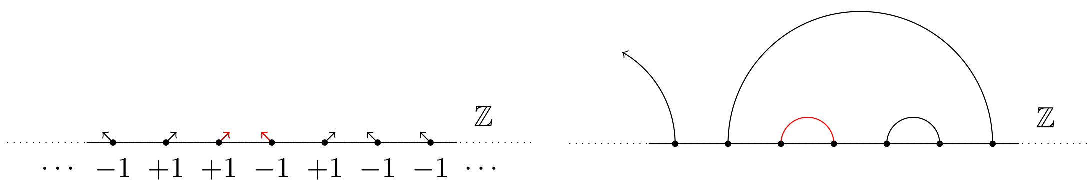


Figure 1.8: Part of a sequence of $\{+1, -1\}$, the corresponding sequence of arrows, and the corresponding (part of) noncrossing matching. In red, two arrows that are necessarily connected, and the corresponding arc.

The proof that this construction a.s. provides a matching on $\mathbb{Z}$ is left to the reader (see Exercise sheet).

We now define the Infinite Noodle Soup.

Definition 1.4.2 (The Infinite Noodle Soup)**.** *The Infinite Noodle Soup is a random variable taking its values in the set $\Omega := \left(\{-1,+1\}^2\right)^{\mathbb{Z}}$. To construct it, consider two independent bi-infinite sequences of i.i.d. Rademacher variables $(\omega_i^+)_{i\in\mathbb{Z}}$ and $(\omega_i^-)_{i\in\mathbb{Z}}$.*

Then, use $(\omega_i^+)_{i\in\mathbb{Z}}$ to construct an infinite noncrossing matching above the axis, and $(\omega_i^-)_{i\in\mathbb{Z}}$ to construct one below the axis. The resulting object is called the Infinite Noodle Soup (INS), see Fig. 1.9.

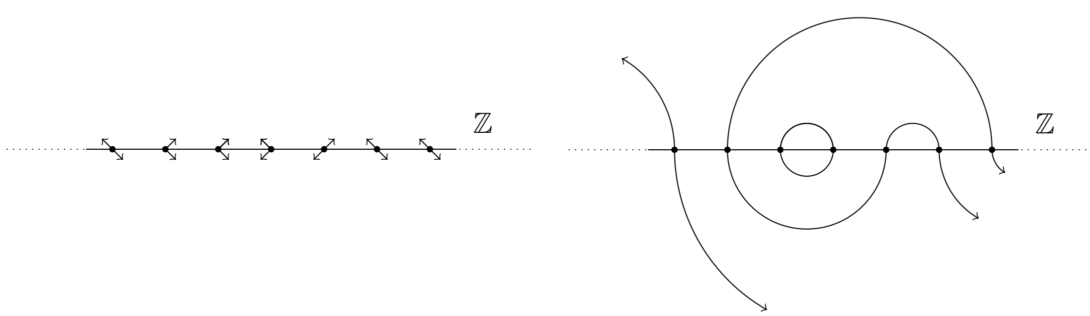


Figure 1.9: A part of the Infinite Noodle Soup, constructed from two bi-infinite sequences of arrows.

It is important to notice that the INS is made of closed loops, plus possibly some infinite paths. We will call the components *loops* when they are closed, and *paths* when they are infinite.

The INS is also sometimes called UIMS (Uniform Infinite Meandric System), e.g. in [2]. It was first introduced and studied by Curien, Kozma, Sidoravicius and Tournier [4], who were interested in the following question.

Question 1.4.3. *Is there an infinite noodle in the Infinite Noodle Soup?*

In other words, does there exist an infinite path?

We will give in Chapter 2 a partial answer to this question, due to [4], which uses arguments adapted from percolation theory.

We can now state our first main result, which is the convergence of a uniform meandric system m_n towards the INS. To this end, let us start by defining the notion of convergence we are interested in. To this end, we will consider marked meandric systems, that is, pairs (M, j) where M is a meandric system on some interval $I \subseteq \mathbb{Z}$ and $j \in I$.

Definition 1.4.4 (Local convergence)**.** *Let $(M_n, j_n)_{n\geq 1}$ be a sequence of marked meandric systems on (possibly infinite) intervals $I_n \subseteq \mathbb{Z}$, with $j_n \in I_n$, and (M_∞, j_∞) a marked meandric system on $I_\infty \subseteq \mathbb{Z}$. We see them as configurations $\left(\omega_i^{+,(n)}, \omega_i^{-,(n)}\right)_{i\in\mathbb{Z}}$ in $(\{+1,-1\}^2)^{I_n} \times I_n$. We say that $((M_n, j_n))_{n\geq 1}$ converges locally to (M_∞, j_∞), and write $(M_n, j_n) \underset{n\to\infty}{\overset{(loc)}{\to}} (M_\infty, j_\infty)$, if, for all $R \geq 0$, the following convergence holds in distribution:*

$$\left(\omega_i^{+,(n)}, \omega_i^{-,(n)}\right)_{j_n-R\leq i\leq j_n+R} \underset{n\to\infty}{\overset{(d)}{\to}} \left(\omega_i^{+,(\infty)}, \omega_i^{-,(\infty)}\right)_{j_\infty-R\leq i\leq j_\infty+R}.$$

In other words, for any configuration π of $(\{+1,-1\}^2)^{2R+1}$, the probability to observe π in M_n around j_n converges to the probability to observe π in M_∞ around j_∞.

Theorem 1.4.5. *Let m_n be a uniform meandric system on $\{1, \ldots, 2n\}$, $j_n \in \{1, \ldots, 2n\}$ be uniform and independent of m_n. Let m_∞ be the INS on $\mathbb{Z}$. Then, we have*

$$(m_n, j_n) \underset{n\to\infty}{\overset{(loc)}{\to}} (m_\infty, 0).$$

In other words, for any fixed $R \geq 0$, the probability that we observe a given element $(\{+1,-1\}^2)^{2R+1}$ in m_n between $j_n - R$ and $j_n + R$, converges to $\frac{1}{4^{2R+1}}$.

To prove it, we will consider the above and below parts separately, and rephrase Theorem 1.4.5 using the representation of noncrossing matchings as Dyck paths.

Proof. A configuration of $\{+1,-1\}$ of length $2R+1$ corresponds to a walk of length $2R+1$ with steps in $\{(+1,+1),(+1,-1)\}$. The proof of Theorem 1.4.5 translates as follows, see Fig. 1.10.

Lemma 1.4.6. *Let d_n be a random uniform Dyck path of semi-length n. Let j_n be uniform on $\{1, \ldots, 2n\}$, independent of d_n. Then, for all $\eta \in \{-1,+1\}^{2R+1}$,*

$$\mathbb{P}\left((d_n(i+1) - d_n(i))_{j_n-R\leq i\leq j_n+R} = \eta\right) \underset{n\to\infty}{\to} \frac{1}{2^{2R+1}}.$$

First, let us prove it for $R = 0$. Let $\eta = \{+1\}$. We want to prove that, for j_n uniform in $\{0, \ldots, 2n-1\}$ independent of the random Dyck path d_n,

$$q_n := \mathbb{P}(d_n(j_n+1) - d_n(j_n) = +1) \to \frac{1}{2}. \tag{1.4.1}$$

Since d_n has exactly n $(+1,+1)$ steps and n $(+1,-1)$ steps, we actually get immediately $\mathbb{P}(d_n(j_n+1) - d_n(j_n) = +1) = \frac{1}{2}$.

This nice argument however breaks whenever $R \geq 1$. To overcome it, let us prove (1.4.1) again for $\eta = \{+1\}$ with another more robust method.

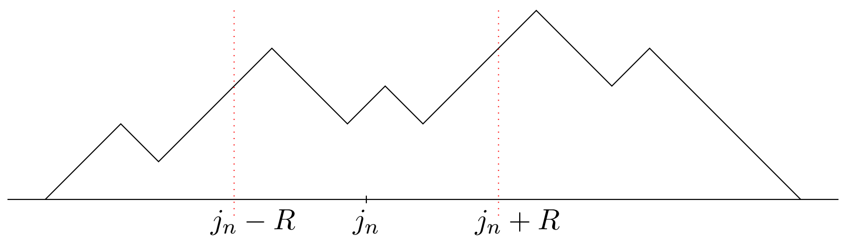


Figure 1.10: A Dyck path, and the R-neighbourhood of a uniform point $j_n \in \{1, \dots, 2n\}$.

Observe that we can write

$$q_n = \frac{1}{(\#\text{ matchings})} \frac{1}{2n} \sum_{j=0}^{2n-1} \sum_{p=0}^{n} S_{(0,0)\to(j,p)} S_{(j+1,p+1)\to(2n,0)} \mathbb{1}_{j=p \mod 2}.$$

Here, $j = j_n$ is the uniform point on $\{1, \dots, 2n\}$, and $p = d_n(j)$. We recall that $S_{a\to b}$ is the number of paths from a to b that stay (weakly) above the horizontal axis.

Observe by symmetry that $S_{(j+1,p+1)\to(2n,0)} = S_{(0,0)\to(2n-j-1,p+1)}$. Using the result of Proposition 1.3.2, we therefore obtain:

$$q_n = \frac{n+1}{\binom{2n}{n}} \frac{1}{2n} \sum_{j=0}^{2n-1} \sum_{p=0}^{n} \frac{p+1}{j+1} \binom{j+1}{\frac{j-p}{2}} \frac{p+2}{2n-j} \binom{2n-j}{\frac{2n-j-p-2}{2}} \mathbb{1}_{j=p \mod 2}.$$

We will only consider the partial sum on Dyck paths such that $j \in [\varepsilon n, (2-\varepsilon)n]$ and $p \in [\varepsilon\sqrt{n}, M\sqrt{n}]$ for $0 < \varepsilon < M$. Let $q_n(\varepsilon, M)$ be the associated truncated sum. Using Stirling's approximation, we get that, for $j = un$ and $p = v\sqrt{n}$, $j = p \mod 2$, uniformly in $c \in [\varepsilon, 2-\varepsilon]$ and in $d \in [\varepsilon, M]$:

$$\begin{aligned} \frac{1}{\binom{2n}{n}} \frac{p+1}{j+1} \binom{j+1}{\frac{j-p}{2}} \frac{p+2}{2n-j} \binom{2n-j}{\frac{2n-j-p-2}{2}} &\sim \frac{d^2}{c(2-c)n} \frac{\sqrt{\pi n}}{4^n} 2^{un+1} \sqrt{\frac{2}{\pi u n}} e^{-\frac{v^2}{2u}} 2^{(2-u)n} \sqrt{\frac{2}{\pi(2-u)n}} e^{-\frac{v^2}{2(2-u)}} \\ &\sim \frac{4v^2}{\sqrt{\pi u^3 (2-u)^3 n^3}} e^{-v^2\left(\frac{1}{2u} + \frac{1}{2(2-u)}\right)} \\ &= \frac{4v^2}{\sqrt{\pi u^3 (2-u)^3 n^3}} e^{-\frac{v^2}{u(2-u)}}. \end{aligned}$$

Therefore, we get

$$q_n \geq q_n(\varepsilon, M) = \frac{n+1}{2n} \frac{1}{2} \int_{u=\varepsilon}^{2-\varepsilon} \int_{v=\varepsilon}^{M} \frac{4v^2}{\sqrt{\pi u^3 (2-u)^3 n^3}} e^{-\frac{v^2}{u(2-u)}} dv du + o(1),$$

where the factor $\frac{1}{2}$ comes from the condition $j = p \mod 2$.

Therefore, taking the liminf as $\varepsilon \to 0$, $M \to \infty$ on both sides, we get

$$\liminf_{n\to\infty} q_n \geq \frac{1}{4}\int_{u=0}^{2}\int_{v=0}^{\infty} \frac{4v^2}{\sqrt{\pi u^3(2-u)^3 n^3}} e^{-\frac{v^2}{u(2-u)}} dv du.$$

Furthermore, using the formula for the second moment of a Gaussian variable

$$\int_{-\infty}^{\infty} v^2 e^{-\lambda v^2} dv = \frac{1}{4}\sqrt{\frac{\pi}{\lambda^3}},$$

we get:

$$\int_{u=0}^{2}\int_{v=0}^{\infty} \frac{4v^2}{\sqrt{\pi u^3(2-u)^3}} e^{-\frac{v^2}{u(2-u)}} dv du = 4\int_{u=0}^{2} \frac{1}{4} du = 2.$$

Finally, we have

$$\liminf_{n\to\infty} \mathbb{P}(d_n(j_n+1) - d_n(j_n) = +1) \geq \frac{1}{2}.$$

But we can do the exact same computation for $\eta = \{-1\}$, and get:

$$\liminf_{n\to\infty} \mathbb{P}(d_n(j_n+1) - d_n(j_n) = -1) \geq \frac{1}{2}.$$

Summing these two inequalities, we get that they must actually be equalities, so that $\mathbb{P}(d_n(j_n+1) - d_n(j_n) = +1), \mathbb{P}(d_n(j_n+1) - d_n(j_n) = -1)$ both converge to $\frac{1}{2}$.

This argument can be generalized in a straightforward way to any value $R \geq 0$, which ends the proof of Lemma 1.4.6, and therefore the proof of Theorem 1.4.5. □

1.5 Number of loops

We can finally prove Theorem 1.2.5. Let m_∞ be the INS. Denote by $\mathcal{C}(0)$ the loop containing 0 in the INS, and by $|\mathcal{C}(0)|$ its size (that is, the number of vertices in the loop $\mathcal{C}(0)$).

Recall that, for m a meandric system, $\mathcal{L}(m)$ denotes the number of loops in m.

Theorem 1.5.1. *We have the following result:*

$$\frac{\mathbb{E}\left[\mathcal{L}(m_n)\right]}{n} \underset{n\to\infty}{\to} \mathbb{E}\left[\frac{2}{|\mathcal{C}(0)|}\right].$$

The value $\mathbb{E}\left[\frac{2}{|\mathcal{C}(0)|}\right]$ that appears is the constant κ of Theorems 1.2.5 and 1.2.6.

The proof uses the local convergence of m_n towards the INS, as well as the following lemma. This lemma, which is deterministic, shows that the number of loops of a meandric system can be expressed as the expectation of a quantity that depends on a small neighbourhood of a uniform vertex.

Lemma 1.5.2. *Let m be any meandric system on $\{1, \dots, 2n\}$. Then, we have*

$$\mathcal{L}(m) = \sum_{i=1}^{2n} \frac{1}{|\mathcal{C}_m(i)|},$$

where $\mathcal{C}_m(i)$ is the loop of m containing the point i. In particular,

$$\mathcal{L}(m) = 2n\,\mathbb{E}\left[\frac{1}{|\mathcal{C}_m(j_n)|}\right],$$

where j_n is uniform on $\{1, \ldots, 2n\}$.

Proof of Lemma 1.5.2. In the sum on the right-hand side, each loop C appears exactly $|C|$ times. We can therefore rewrite

$$\begin{aligned}\sum_{i=1}^{2n} \frac{1}{|\mathcal{C}_m(i)|} &= \sum_{C \text{ loop of } m} \sum_{i \in C} \frac{1}{|\mathcal{C}_m(i)|} \\ &= \sum_{C \text{ loop of } m} \sum_{i \in C} \frac{1}{|C|} \\ &= \sum_{C \text{ loop of } m} 1 \\ &= |\mathcal{L}(m)|.\end{aligned}$$

The second part follows immediately. □

This implies that $\frac{\mathcal{L}(m_n)}{n} = 2\mathbb{E}\left[\frac{1}{|\mathcal{C}_{m_n}(j_n)|}|m_n\right]$ (recall that m_n is a uniform meandric system of size n, independent of j_n), so by taking expectations on both sides we get:

$$\frac{\mathbb{E}[\mathcal{L}(m_n)]}{n} = \mathbb{E}\left[\frac{2}{|\mathcal{C}_{m_n}(j_n)|}\right].$$

Proof of Theorem 1.5.1. We have managed to express $\mathcal{L}(m_n)$ as a local quantity (that is, which depends on neighbourhoods of a uniform point), which is exactly the kind of quantities that are considered by the local convergence. The only thing left to prove is that we can "see" the quantity $\frac{1}{|\mathcal{C}_{m_n}(j_n)|}$ on finite neighbourhoods of j_n.

To this end, observe that we can write

$$\begin{aligned}\mathbb{E}\left[\frac{1}{|\mathcal{C}(0)|}\right] &= \mathbb{E}\left[\frac{1}{|\mathcal{C}(0)|}\mathbb{1}_{|\mathcal{C}(0)|<\infty}\right] \\ &= \sum_{k\geq 1}\mathbb{E}\left[\frac{1}{|\mathcal{C}(0)|}\mathbb{1}_{|\mathcal{C}(0)|=k}\right] \\ &= \lim_{R\to\infty}\sum_{k\geq 1}\frac{1}{k}\mathbb{E}\left[\mathbb{1}_{|\mathcal{C}(0)|=k}\mathbb{1}_{E_R}\right],\end{aligned}$$

where E_R is the event that $\mathcal{C}(0)$ in m_∞ is contained inside $[-R, R]$. Now, for any k, R, the map $(m, j) \mapsto \mathbb{1}_{|\mathcal{C}(j)|=k}\mathbb{1}_{E_R}$ only depends on the R-neighbourhood of j in m, and is bounded (by 1). Hence, by the local convergence of (m_n, j_n) to $(m_\infty, 0)$ (Theorem 1.4.5), we get the result. □

Remark 1.5.1. *Actually, in [8], we prove the (stronger) quenched local convergence of a uniform meandric system to the infinite noodle. Namely, we show that, for any continuous function F, we have:*

$$\mathbb{E}[F(m_n, j_n)|m_n] \underset{n\to\infty}{\to} \mathbb{E}[F(m_\infty, 0)].$$

This implies Theorem 1.2.6, that is, the convergence in probability of $\frac{1}{n}\mathcal{L}(m_n)$ *towards the constant* κ*.*

Chapter 2

Uniqueness of the infinite noodle?

This second chapter is devoted to the detailed study of the INS m_∞. We aim to provide a (partial) proof of the main result of [4], Theorem 2.0.1. Observe that the INS is made of a collection of finite loops, and possibly several infinite paths. Let $n_{paths} \in \{0, 1, \dots, \} \cup \{\infty\}$ denote the number of infinite paths in m_∞. Then, we have the following result.

Theorem 2.0.1 (Uniqueness of the infinite noodle [4])**.** *Either $n_{paths} = 0$ almost surely, or $n_{paths} = 1$ almost surely.*

In other words, if there exists an infinite noodle, it is unique a.s. The authors strongly expect that $n_{paths} = 0$ almost surely, that is, there is no infinite path in m_∞, but it is still an open problem.

We will only prove here a weaker version of this theorem, and will not rule out the case $n_{paths} = 2$. Namely, we will show:

Theorem 2.0.2 (Uniqueness of the infinite noodle [4])**.** *Either $n_{paths} = 0$ almost surely, or $n_{paths} = 1$ almost surely, or $n_{paths} = 2$ almost surely.*

2.1 n_{paths} is constant

A first step in the proof of Theorem 2.0.2 is to prove the following lemma.

Lemma 2.1.1. *There exists $p \in \{0, 1, \dots\} \cup \{\infty\}$ such that, almost surely, $n_{paths} = p$. In other words, n_{paths} is almost surely constant.*

2.1.1 Ergodic theory

The proof of Lemma 2.1.1 relies on Birkhoff's *ergodic theorem*, which we will now recall.

Definition 2.1.2. *Let $(\Omega, \mathcal{A}, \mu)$ be a measured space.*

- *We say that a measurable transformation $T : \Omega \to \Omega$ is measure-preserving if, for all $A \in \mathcal{A}$:*

$$\mu(T^{-1}(A)) = \mu(A).$$

- *We say that a measure-preserving transformation T is ergodic if, for any $A \in \mathcal{A}$ such that $T(A) = A$, we have*

$$\mu(A) \in \{0, 1\}.$$

The interest of this definition lies in the following theorem, due to Birkhoff.

Theorem 2.1.3 (Birkhoff's ergodic theorem (1931))**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a measured space, $T : \Omega \to \Omega$ be a measure-preserving transformation, and $f \in L^1(\Omega)$. Then, we have that, for almost all $\eta \in \Omega$,*

$$\frac{1}{n}\sum_{k=1}^{n} f \circ T^k(\eta)$$

converges to some (random) value $g(\eta)$. If, in addition, T is ergodic, then $g(\eta)$ is constant a.s., equal to $\mathbb{E}_\mu[f(\eta)]$.

2.1.2 Application to the infinite noodle

We can apply it to our model. Consider $\Omega = \left((\{-1,1\})^2\right)^{\mathbb{Z}}$, our set of infinite configurations (again, recall that $+1$ corresponds to an arc going to the right, and -1 to an arc going to the left). Let $\mathcal{A}$ be the σ-algebra generated by the cylinders (that is, finitely many of the ω_x^+, ω_x^-). Let $\mu := \otimes^{\mathbb{Z}} (Ber(1/2) \otimes Ber(1/2))$ be the probability measure on Ω associated to the INS. We consider the transformation $T : \Omega \to \Omega$ defined as:

$$T\left(\left(\omega_n^+, \omega_n^-\right)_{n\in\mathbb{Z}}\right) = \left(\omega_{n+1}^+, \omega_{n+1}^-\right)_{n\in\mathbb{Z}},$$

which corresponds to a shift of length 1 to the right. Then, T is clearly measure-preserving, and it can be shown that it is also ergodic. The ergodicity of the Bernoulli shift is a classical result, and will not be proved in these lecture notes.

Proof of Lemma 2.1.1. Consider $f_p : \Omega \to \mathbb{R}, \eta \mapsto \mathbb{1}_{\eta \text{ has } p \text{ infinite paths}}$. Then f_p is L_1, so by Birkhoff's ergodic theorem, $\frac{1}{n}\sum_{k=1}^{n} f_p \circ T^k(\eta)$ converges a.s. to some constant c. Since $f_p(\eta) = f_p(T(\eta))$, we have that, for all $n \geq 1$, all $\eta \in \Omega$,

$$\frac{1}{n}\sum_{k=1}^{n} f_p \circ T^k(\eta) = f_p(\eta).$$

Hence, $c = 0$ a.s. or $c = 1$ a.s. Since this is true for all $p \in \{0, 1, \dots\} \cup \{\infty\}$, necessarily there exists exactly one value of p such that $n_{paths} = p$ a.s. □

2.2 A trifurcation argument

In the rest of the section, our goal is to show that $n_{paths} \leq 2$ a.s. We will not show that $n_{paths} \leq 1$ a.s., and refer the reader to [4, Section 4] for a proof.

The structure of the proof of Theorem 2.0.2 is as follows:

- Step 1: The INS restricted to an interval $[\![a,b]\!]$ is made of closed loops, plus some dangling edges (see Fig. 2.1), which are half-arcs going out of the interval. We denote by $n_d(a,b)$ the (random) number of dangling edges going out of $[\![a,b]\!]$ in the INS. Considering the interval $I_N := [\![1,N]\!]$, for large values of N, we will show:

  Lemma 2.2.1. *The number $n_d(1,N)$ of dangling edges from I_N satisfies*
  $$\frac{n_d(1,N)}{N} \underset{N\to\infty}{\overset{a.s.}{\to}} 0.$$

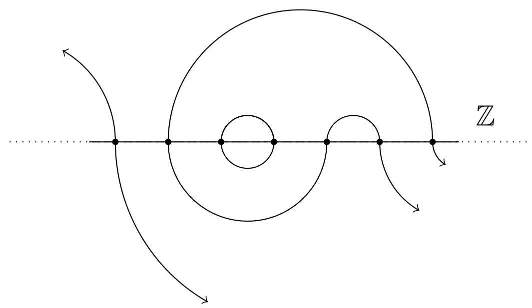


Figure 2.1: A part of the infinite noodle, with one closed loop and 4 dangling edges.

- Step 2: we will define a set Tri of special points in the INS, which we call *trifurcations* (see Definition 2.2.5). We have the following result:

  Lemma 2.2.2. *If $n_{paths} \geq 3$ a.s., then $\mathbb{P}(0 \in Tri) > 0$.*

- Step 3: we will prove that the number of trifurcations in I_N is smaller than a constant times the number of dangling edges.

  Lemma 2.2.3. *Assume that $n_{paths} \geq 3$. For all odd N large enough, we have*
  $$\#\left(I_N \cap Tri\right) \leq 2\#\{\mathcal{C}, |\mathcal{C}| = \infty, \mathcal{C} \cap [\![-2, N+3]\!] \neq \emptyset\} \leq n_d(-2, N+3).$$

Lemmas 2.2.1,2.2.2 and 2.2.3 actually imply our result.

Proof of Theorem 2.0.2. Let us assume that $n_{paths} \geq 3$ almost surely. Thus, by Lemma 2.2.2, $\mathbb{P}(0 \in Tri) > 0$. Therefore, by Birkhoff's ergodic theorem applied to $f : \eta \mapsto \mathbb{1}_{0 \in Tri}$, we get that
$$\frac{\#\left(I_N \cap Tri\right)}{N} \underset{N\to\infty}{\overset{a.s.}{\to}} \mathbb{P}(0 \in Tri) > 0. \tag{2.2.1}$$
Hence, $\#\left(I_N \cap Tri\right)$ grows linearly with N.

On the other hand, the invariance by translation of the model shows that, for all N, for all N odd, $n_d(-2, N+3) \overset{(d)}{=} n_d(1, N+6)$. Therefore, Lemmas 2.2.1 and 2.2.3 together show that
$$\frac{\#\left(I_N \cap Tri\right)}{N} \underset{N\to\infty}{\overset{a.s.}{\to}} 0,$$
which contradicts (2.2.1). The result follows from Lemma 2.1.1. $\square$

We can now focus on proving our three lemmas.

2.2.1 Dangling edges

We start by showing that the number of dangling edges in I_N is small compared to N.

Proof of Lemma 2.2.1. □

Consider the restriction of the INS to $I_N := [\![1, N$. Recall the notation ω_x^+, ω_x^- for $x \in \mathbb{Z}$, and set, for $1 \leq k \leq N$, $S_k^+ = \omega_1^+ + \ldots + \omega_k^+$ and $S_k^- = \omega_1^- + \ldots + \omega_k^-$. Then, S^+, S^- are two independent centered random walks. Furthermore, notice that

$$n_d(N) \leq \max_{1\leq k\leq N} S_k^+ - \min_{1\leq k\leq N} S_k^+ + \max_{1\leq k\leq N} S_k^- - \min_{1\leq k\leq N} S_k^-.$$

Indeed, a dangling edge going to the left above the axis corresponds to a local minimum of S^+, and similar statements hold for dangling edges going to the left or right, above or below the axis.

Hence, it follows from the law of large numbers that $\frac{n_d(N)}{N} \to 0$ almost surely as $N \to \infty$, since $\lim\limits_{N\to\infty} \frac{S_N^+}{N} = \lim\limits_{N\to\infty} \frac{S_N^-}{N} = 0$.

2.2.2 Trifurcations

We define here the concept of trifurcations, which may be reminiscent of trifurcations in the context of infinite clusters of percolation. Roughly speaking, a *trifurcation* is an integer which is part of an infinite path, and close to two other infinite paths. In particular, there may exist a trifurcation only when $n_{paths} \geq 3$.

Definition 2.2.4. *Let $x \in \mathbb{Z}$. We define $\mathcal{C}^+(x)$ as the first infinite path above x, if it exists. Specifically, $\mathcal{C}^+(x) = \mathcal{C}(x')$, where:*

$$x' := \inf\{y > x; \exists z < x, y \leftrightarrow z, |\mathcal{C}(y)| = \infty\}.$$

We define the same way $\mathcal{C}^-(x)$.

Definition 2.2.5. *We say that $x \in \mathbb{Z}$ is a trifurcation if the following hold:*

(i) $|\mathcal{C}(x)| = |\mathcal{C}^+(x)| = |\mathcal{C}^-(x)| = \infty$*;*

(ii) $(\omega_x^+, \omega_x^-) \in \{(-1,-1), (1,1)\}$*;*

(iii) $d(x, \mathcal{C}^+(x)) \leq 3, d(x, \mathcal{C}^-(x)) \leq 3$.

(iv) $\mathcal{C}(x), \mathcal{C}^+(x), \mathcal{C}^-(x)$ *are disjoint.*

Here the distance d is the natural distance in $\mathbb{Z}$. Although this definition looks unnatural, an important consequence is that, if x is a trifurcation, then among the three infinite clusters $\mathcal{C}(x), \mathcal{C}^+(x), \mathcal{C}^-(x)$, none of the three separates the other two.

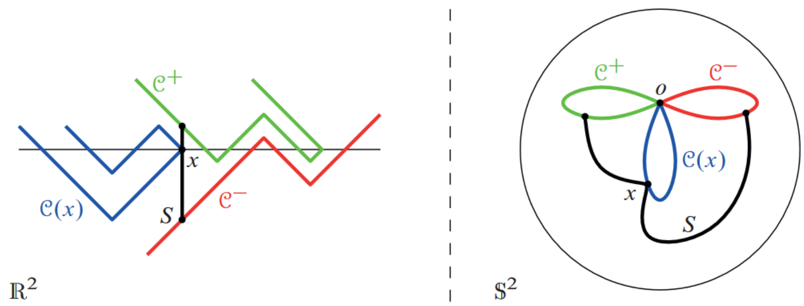


Figure 2.2: Left: an example of a trifurcation. Right: an embedding in $\mathbb{S}^2$. Figure taken from [4].

2.2.3 If the number of paths is too big...

We will show here Lemma 2.2.2, which states that, if we have at least three infinite components, then we have $\mathbb{P}(0 \in Tri) > 0$.

Proof of Lemma 2.2.2. We will prove that, if we have three infinite paths close to each other, we can rewire them inside $[\![-M, M]\!]$ for M large to obtain a trifurcation at 0. We will only sketch the proof, and refer to [4] for details.

Assume that $n_{paths} \geq 3$ almost surely. Then, there exists $M \geq 1$ large enough such that $\mathbb{P}(A_M) > 0$, where A_M is the event that three infinite paths cross $[\![-M, M]\!]$.

Now, we consider such an M and three such infinite paths, and show that we can rewire everything inside $[\![-M, M]\!]$ to create a trifurcation at 0. For any $\eta \in A_M$, let $\tilde{\eta}$ be a configuration obtained by keeping η on $[\![-M, M]\!]^c$, and rewiring it inside $[\![-M, M]\!]$ to create a trifurcation at 0, and define $\tilde{A}_M := \{\tilde{\eta}; \eta \in A_M\}$. We claim that such a configuration exists, i.e. it is always possible to rewire inside $[\![-M, M]\!]$.

Then, clearly, $\mathbb{P}(\tilde{A}_M) \geq 2^{-2(2M+1)}\mathbb{P}(A_M) > 0$, as one changes at most $2(2M+1)$ variables between a configuration η and the modified configuration $\tilde{\eta}$, namely $\{(\omega_x^+, \omega_x^-)_{-M \leq x \leq M}\}$.

We now sketch the proof of the rewiring claim. Consider three infinite paths entering and exiting $[\![-M, M]\!]$. We only care about the six entrance/exit points of these paths. See Fig. 2.3, up-left, for one possible way to connect these three points so that none of these three paths separates the other two, where the three paths considered are drawn thicker. One can be convinced that, if one takes three paths with given relative positions, then there is a common way to rewire them inside $[\![-M, M]\!]$, independently of M, and complete $\tilde{\eta}$ afterwards without changing the fact that the three paths make a trisection at 0. Since there are only finitely many relative positions for three paths, we only need to find a way to rewire for each of them. This is fully done in [4]. On Fig. 2.3 is one such example, along with a way of rewiring it.

$\square$

Remark 2.2.1. *The proof written above is wrong per se. In particular, one cannot necessarily guarantee a trifurcation at* 0*, but, for parity reasons, one can guarantee a trifurcation either at* 0 *or at* 1*. The idea of the proof is the asme, and details can be found in [4].*

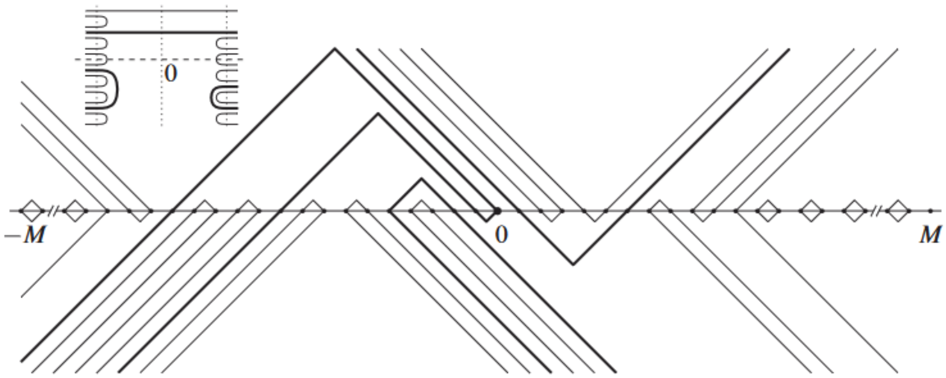


Figure 2.3: An example of rewiring. Figure taken from [4].

2.2.4 There are less trifurcations than infinite paths.

The last part of the proof of Theorem 2.0.2 is to show that there are not many trifurcations (Lemma 2.2.3).

Proof of Lemma 2.2.3. Consider all infinite paths in the INS. After removing the horizontal axis, these paths split the plane into different (unbounded) faces. See e.g. Fig. 2.2, left, where there are three infinite paths and four faces: one delimited by each infinite path, and one inbetween them. One can alternatively see them as faces on $\mathbb{S}^2$, obtained as the natural compactification of $\mathbb{R}^2$, see Fig. 2.2, right. Consider F one of these faces, and a collection $\mathcal{U}$ of infinite paths which are on the boundary ∂F of F (not necessarily all of them). We let $Tri(\mathcal{U})$ be the set of trifurcations x such that $\mathcal{C}(x), \mathcal{C}^+(x), \mathcal{C}^-(x) \in \mathcal{U}$.

The claim, which we will prove by induction, is the following:

$$\text{if } |\mathcal{U}| \geq 2, \text{ then } |Tri(\mathcal{U})| \leq |\mathcal{U}| - 2.$$

Initialization If $|\mathcal{U}| = 2$, it is clear, since a trifurcation requires the coexistence of three disjoint infinite paths.

Induction step Now let $p \geq 2$, and assume that the result holds for $|\mathcal{U}| \leq p$. Fix a face F, and consider $\mathcal{U}$ such that $|\mathcal{U}| = p + 1$. If $Tri(\mathcal{U}) = \emptyset$, then the property holds immediately. Otherwise, let $x \in Tri(\mathcal{U})$. Consider the vertical segment S passing through x, such that its endpoints are the first intersection points with $\mathcal{C}^+(x)$ and $\mathcal{C}^-(x)$. By definition, it does not intersect any other infinite path, and splits F into three disjoint unbounded faces F_1, F_2, F_3, see Fig. 2.2. Define $\mathcal{U}_i = \mathcal{U} \cap \partial F_i$ for all $1 \leq i \leq 3$.

By construction, only $\mathcal{C}(x), \mathcal{C}^+(x)$ and $\mathcal{C}^-(x)$ belong to more than one of the $\mathcal{U}_i$'s, and each one belongs to exactly two. In particular, $2 \leq |\mathcal{U}_i| \leq p$ for $i = 1, 2, 3$, and $|\mathcal{U}_1| + |\mathcal{U}_2| + |\mathcal{U}_3| = |\mathcal{U}| + 3$. Furthermore, for $y \in Tri(\mathcal{U}) \setminus \{x\}$, there exists a unique $i \in \{1, 2, 3\}$ such that $y \in Tri(\mathcal{U}_i)$. By induction, we get

$$|Tri(\mathcal{U})| \leq 1 + \sum_{i=1}^{3} |Tri(\mathcal{U}_i)|$$

$$\leq 1 + \sum_{i-1}^{3}(|\mathcal{U}_i| - 2) = 1 + |\mathcal{U}| + 3 - 6$$
$$= |\mathcal{U}| - 2,$$

and the induction is over. In particular, for any face F, denoting by $\mathcal{U}_F$ the set of all infinite paths in the boundary of F, we have

$$|Tri(\mathcal{U}_F)| \leq |\mathcal{U}_F|. \tag{2.2.2}$$

Indeed, it holds if $|\mathcal{U}_F| = 1$ as such faces do not have any trisection, and it holds if $|\mathcal{U}_F| \geq 2$ by the previous induction.

Let us now sum (2.2.2) over all faces. Since one infinite path belongs to the boundary of at most two faces, we get

$$|Tri| = \sum_{F \text{ face}} |Tri(\mathcal{U}_F)| \leq \sum_{F \text{ face}} |\mathcal{U}_F|$$
$$\leq 2\#\{\mathcal{C}, |\mathcal{C}| = \infty, \mathcal{C} \cap [\![-M, M]\!] \neq \emptyset\},$$

where we used the fact that $\mathcal{C}^+(x), \mathcal{C}^-(x)$ are at distance at most 3 from x, for any $x \in Tri$.

Finally, each infinite path crossing $[\![-M-3, M+3]\!]$ is responsible for exactly two dangling edges. Hence:

$$|Tri| \leq 2\#\{\mathcal{C}, |\mathcal{C}| = \infty, \mathcal{C} \cap [\![-M-3, M+3]\!] \neq \emptyset\}$$
$$\leq n_d(-M-3, M+3).$$

Using again the invariance by translation of the INS and the fact that N is odd, this ends the proof of Lemma 2.2.3. $\square$

Chapter 3

Additional results

In this additional chapter, we gather recent results concerning large meandric systems. The first part is devoted to the (conjectured) scaling limit of a uniform meandric system of size n, as $n \to \infty$, following [2]. The second part, more combinatorial, aims at computing the constant κ appearing in Theorems 1.2.5 and 1.2.6.

3.1 Scaling limit of a uniform meandric system

There are conjectures concerning the so-called *scaling limit* of a meandric system. Let us briefly talk about it.

Definition 3.1.1. *Consider a meandric system m on $\{1, \dots, 2n\}$. Embed it on the sphere, by the natural compactification of $\mathbb{R}^2$ (that is, connecting the two endpoints of the horizontal axis). We endow the set $V_n := \{1, \dots, 2n\}$ with a natural distance d_m, by saying that all arcs of m have length 1, and i and $i+1$ ($1 \leq i \leq 2n$) and $2n$ and 1, are connected by edges of length 1. Therefore, we get from a uniform meandric system m_n a metric space (V_n, d_{m_n}) embedded on the sphere. We see this metric space as decorated by a path P_n (corresponding to the horizontal axis) and a collection of loops Γ_n (corresponding to the loops of the meandric system).*

It is conjectured, see [2], that this metric space (V_n, d_{m_n}) has a scaling limit as $n \to \infty$.

Conjecture 3.1.2. *As $n \to \infty$, under an appropriate scaling, we have that (V_n, P_n, Γ_n) converges in distribution to a limiting decorated space, which consists in three independent random objects:*

- *a $\sqrt{2}$-LQG sphere (Liouville Quantum Gravity), which is a random fractal surface;*
- *a space-filling SLE_8 (Schramm-Loewner Evolution) from ∞ to ∞, which is a random fractal curve;*
- *a CLE_6 (Conformal Loop Ensemble), which is a random countable collection of noncrossing loops.*

LQG was first introduced in [5, 7], SLE in [15] and CLE in [12].

In particular, Borga, Gwynne and Park, based on this conjecture, make another one:

Conjecture 3.1.3. *Let m_n be a uniform meandric system of size n. Then, the largest loop in m_n has a number of vertices of order $n^{\alpha+o(1)}$, where $\alpha = \frac{3-\sqrt{2}}{2}$.*

This is supported by simulations, see [2, Section 7.4].

3.2 Estimates on the probability of 0 being in an infinite component

The goal of this last part is to prove partial results about the constant κ. Recall from Theorem 1.5.1 that we have the expression

$$\kappa = \mathbb{E}\left[\frac{2}{|\mathcal{C}(0)|}\right],$$

where $\mathcal{C}(0)$ is the component (loop or path) containing 0 in m_∞. Therefore, we can write

$$\begin{aligned}\kappa &= 2\sum_{k\geq 1}\frac{1}{2k}\mathbb{P}\left(|\mathcal{C}(0)| = 2k\right)\\ &= \sum_{k\geq 1}\frac{1}{k}\mathbb{P}\left(|\mathcal{C}(0)| = 2k\right).\end{aligned}$$

Hence, in order to compute the value of κ, one would like to compute the probability that $\mathcal{C}(0)$ has a given size $2k$. With A. Bostan and V. Féray [3], we recently proved the following result.

Theorem 3.2.1. *For any $k \geq 1$,*

$$\mathbb{P}\left(|\mathcal{C}(0)| = 2k\right) \in \mathbb{Q}\left[\frac{1}{\pi}\right],$$

that is, it can be written as a polynomial with rational coefficients in $\frac{1}{\pi}$.

Furthermore, for any k fixed, we provide an algorithm that can explicitly compute the associated probability. The question of whether a combinatorial miracle could occur, and allow us to compute κ by summing over all $k \geq 1$, is still open.

Remark 3.2.1. *In another direction, the probability that 0 belongs to an infinite path can be written*

$$1 - \sum_{k\geq 1}\mathbb{P}\left(|\mathcal{C}(0)| = 2k\right).$$

This provides another motivation to the computation of these quantities, as this probability is 0 if and only if $n_{paths} = 0$ a.s..

The proof of Theorem 3.2.1 is intricate and technical, and based on an induction argument on a set of trees. In what follows, we will provide a proof for small values of k.

The case $k = 1$ Consider the smallest possible value of k: $k = 1$. In this case, $\mathcal{C}(0)$ is just a small circle, that is, 0 is connected to some integer $2j + 1 (j \in \mathbb{Z})$, above and below the axis. Inside this circle, one must have two noncrossing matchings on $\{1, \dots, 2j\}$ or $\{2j + 2, \dots, -1\}$ (depending on the sign of j), one above and one below the axis. Therefore, we obtain the nice expression:

$$\mathbb{P}\left(|\mathcal{C}(0)| = 2\right) = 2 \sum_{j \geq 0} Cat_j^2 \left(\frac{1}{2}\right)^{4j+4}.$$

Indeed, if one fixes j and the noncrossing matchings inside the circle, the probability that this configuration occurs is exactly $\left(\frac{1}{2}\right)^{4j+4}$, as $4j + 4$ of the ω_i^+, ω_i^- are fixed. Hence, we only have to compute

$$\sum_{j \geq 0} Cat_j^2 16^{-j}.$$

Proposition 3.2.2. *We have the equality*

$$S := \sum_{j \geq 0} Cat_j^2 16^{-j} = \frac{16}{\pi} - 4.$$

To prove it, we will consider special functions called hypergeometric functions. These functions were first studied by Gauss, who discovered many relations between them. Hypergeometric functions can be thought of as series $\sum_{n \geq 0} a_n z^n$ with positive radius of convergence, such that the ratio of two consecutive coefficients a_{n+1}/a_n, is a rational function of n. We refer to [1] for an extensive study of special functions - including hypergeometric functions - and their properties.

Definition 3.2.2. *For $a, b, c \in \mathbb{R}$ (with $-c \notin \mathbb{N}$), the hypergeometric function ${}_2F_1(a, b; c; z)$ is defined as*

$${}_2F_1(a, b; c; z) = \sum_{n \geq 0} \frac{a^{\uparrow n} b^{\uparrow n}}{c^{\uparrow n}} \frac{z^n}{n!},$$

where $a^{\uparrow n} = a(a + 1) \dots (a + n - 1)$ is the ascending factorial.

It turns out that the infinite sum S rewrites in terms of a hypergeometric function, namely:

Lemma 3.2.3. *We have*

$$\begin{aligned} S &= -4 + 4\, {}_2F_1(-\frac{1}{2}, -\frac{1}{2}; 1; 1) \\ &= -4 + 4 \sum_{n \geq 0} \frac{\left((-\frac{1}{2})^{\uparrow n}\right)^2}{n!^2}. \end{aligned}$$

Assuming Lemma 3.2.3, we can now invoke a result of Gauss.

Theorem 3.2.4 (Gauss' summation identity)**.** *The following identity holds for $c-a-b>0$ and $-c \notin \mathbb{N}$:*

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. $$

We can apply it to prove Proposition 3.2.2.

Proof of Proposition 3.2.2. By Lemma 3.2.3, we have

$$ \begin{aligned} S &= 4\,{}_2F_1(-\frac{1}{2},-\frac{1}{2};1;1)-4 \\ &= 4\,\frac{\Gamma(1)\Gamma(2)}{\Gamma(3/2)^2}-4, \end{aligned} $$

by Theorem 3.2.4. It is known that $\Gamma(1)=\Gamma(2)=1$, and $\Gamma(3/2)=\frac{\sqrt{\pi}}{2}$. Hence, we get

$$ \begin{aligned} S &= 4\frac{1}{(\sqrt{\pi}/2)^2}-4 \\ &= \frac{16}{\pi}-4. \end{aligned} $$

□

The only thing left is to prove Lemma 3.2.3.

Proof of Lemma 3.2.3. Define $u_0=1$ and, for all $n\geq 1$, $u_n=4^{1-2n}Cat_{n-1}^2$. Then we have $(n-1/2)^2u_n=(n+1)^2u_{n+1}$ for all $n\geq 0$. Indeed, for $n=0$ this rewrites $1/4=1/4Cat_0^2$, which is true. For $n\geq 1$, we have

$$ \begin{aligned} \frac{u_{n+1}}{u_n} &= \frac{1}{16}\left(\frac{Cat_n}{Cat_{n-1}}\right)^2 \\ &= \frac{1}{16}\left(\frac{n}{n+1}\right)^2\left(\frac{\binom{2n}{n}}{\binom{2n-2}{n-1}}\right)^2. \end{aligned} $$

Since $\frac{\binom{2n}{n}}{\binom{2n-2}{n-1}}=\frac{2n(2n-1)}{n^2}=\frac{4(n-1/2)}{n}$, we get

$$ \begin{aligned} \frac{u_{n+1}}{u_n} &= \frac{1}{16}\frac{n^2}{(n+1)^2}\times 16\frac{(n-1/2)^2}{n^2} \\ &= \frac{(n+1)^2}{(n-1/2)^2}. \end{aligned} $$

This immediately implies that

$$ \begin{aligned} \sum_{n\geq 0} u_n &= \frac{\left((-\frac{1}{2})^{\uparrow n}\right)^2}{n!^2} \\ &=_2 F_1(-\frac{1}{2},-\frac{1}{2};1;1). \end{aligned} $$

Finally, observe that

$$S = 4\sum_{n\geq 1} u_n = 4\left({}_2F_1(-\frac{1}{2}, -\frac{1}{2}; 1; 1) - u_0\right).$$

We conclude using the fact that $u_0 = 1$. □

Chapter 4

Exercises

4.1 Meandric systems

Exercise 1 (Exponential growth of the number of meanders)**.** *Our goal here is to prove that, as $n \to \infty$, we have*

$$\mathcal{A}_n^{\frac{1}{n}} \to \rho$$

for some $\rho > 0$, where $\mathcal{A}_n$ is the number of meanders of size n.

(i) Consider $p, q \geq 1$. Let $m^{(1)}, m^{(2)}$ be two meanders of respective sizes p, q. Consider the meandric system ms of size $p+q$ obtained by concatenating $m^{(1)}$ and $m^{(2)}$, that is, drawing $m^{(1)}$ between points 1 and $2p$, and $m^{(2)}$ between points $2p+1$ and $2p+2q$. Prove that one can construct a meander of size $p+q$ from ms, by only changing the way $2p$ and $2p+1$ are matched in the bottom noncrossing matching of ms.

(ii) Deduce from it that, for all $p, q \geq 1$,

$$\mathcal{A}_{p+q} \geq \mathcal{A}_p \mathcal{A}_q.$$

(iii) Conclude.

Exercise 2 (Dyck prefixes)**.** *We want here to prove Proposition 1.3.2. Let $i \geq j \geq 0$ be such that $i + j$ is even.*

(i) Show that the number of walks from $(0,0)$ to (i,j) (not necessarily nonnegative) with steps ± 1 is given by $\binom{i}{\frac{i+j}{2}}$.

(ii) Show that the number of such walks that are negative at some point, is the same as the number of walks from $(0,0)$ to $(i,-j-2)$. Hint: consider the first moment such a walk hits -1, and use a symmetry argument.

(iii) Conclude and prove Proposition 1.3.2.

Exercise 3 (Noncrossing partitions and meandric systems)**.** *An integer n being fixed, we define an integer partition as follows. It is a partition of the set $\{1,\dots,n\}$ into nonempty blocks, such that there are no $i<j<k<\ell$ for which i,k are in the same block and j,ℓ are in the same (different) block. We define $NCP(n)$ as the set of noncrossing partitions of n.*

We can now define an order relation $\preceq$ on $NCP(n)$ as follows. We say that $\pi \preceq \sigma$ ($\pi,\sigma \in NCP(n)$) if each block of σ is a union of blocks of π.

(i) Prove that there is a global minimum and a global maximum on $NCP(n)$ for the relation $\preceq$.

(ii) Prove that for any $\pi,\sigma \in NCP(n)$, the set $\{\tau | \tau \preceq \pi, \tau \preceq \sigma\}$ has a maximum for $\preceq$, which we denote $\pi \wedge \sigma$. We define the same way $\pi \vee \sigma := \min\{\tau \in NCP(n) | \pi \preceq \tau, \sigma \preceq \tau\}$.

This fact makes $(NCP(n), \preceq)$ a lattice.

We can now define the Hasse diagram $\mathcal{H}_n$ of $NCP(n)$ as the graph whose vertices are the partitions of $NCP(n)$, and where π is connected to σ if $\pi \neq \sigma$, $\pi \preceq \sigma$, and there is no $\tau \in NCP(n)\backslash\{\pi,\sigma\}$ such that $\pi \preceq \tau \preceq \sigma$.

(iii) Draw the Hasse diagram of $NCP(3)$.

The paper [9] shows the following:

Theorem 4.1.1. *There exists a bijection ϕ between $NCP(n)^2$ and the set$\mathcal{M}_n$ of meandric systems of size n such that, for $(\pi,\sigma) \in NCP(n)^2$, the number of components in the meandric system $\phi(\pi,\sigma)$ is equal to $n - d_{\mathcal{H}_n}(\pi,\sigma)$, where $d_{\mathcal{H}_n}$ is the graph distance in $\mathcal{H}_n$.*

(iv) Show from this theorem that, if π_n, σ_n are i.i.d. uniform elements of $NCP(n)$, then

$$\frac{1}{n} d_{\mathcal{H}_n}(\pi_n,\sigma_n) \underset{n\to\infty}{\to} 1-\kappa,$$

for some $\kappa \in (0,1)$.

(v) Find a bijection between $\mathcal{M}_n$ and $NCP(n)$, and prove the theorem above.

4.2 The infinite noodle soup

Exercise 4 (The INS is a matching)**.** *The goal of this exercise is to prove that the INS is well-defined. In other words, almost surely, for all $i \in \mathbb{Z}$, i is connected to an integer j above and below the axis.*

(i) For $j \in \mathbb{Z}$, let p_j be the probability that 0 is connected to j above the axis in $\mathcal{M}_\infty$. Prove that, for all $k \geq 0$, $p_{2k} = 0$ and

$$p_{2k+1} = 2^{2k-2} Cat_k.$$

(ii) Let $F : x \mapsto \sum_{k\geq 0} Cat_k x^k$. Prove that the radius of convergence of F is $1/4$ and that, for all $x \in [0,1/4]$, we have

$$F(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

(iii) *Deduce that* $\sum_{k\geq 0} p_k = \frac{1}{2}$.

(iv) *Conclude that* 0 *is a.s. connected by an arc.*

(v) *Conclude that* $\mathcal{M}_\infty$ *is an infinite noncrossing matching.*

Exercise 5. *Let* m_∞ *be the infinite noodle soup. We want to prove that the infinite path, if it exists, is actually bi-infinite, that is, goes arbitrarily far from* 0 *in both directions. Hence, we will assume here that* $n_{paths} = 1$ *a.s.*

(i) *Assume that, with positive probability, the infinite path is only infinite in one direction. Show that, with positive probability, its leftmost point is* 1.

(ii) *Show that, in that case, with positive probability there would exist two infinite paths.*

(iii) *Conclude.*

4.3 Catalan sums

$M_{2,1}$

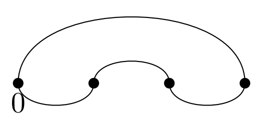


Figure 4.1: The bean meander

Exercise 6. *Consider* M *the meander of Fig. 4.1. We say that the component* $\mathcal{C}(0)$ *of* 0 *in the INS has shape* M *if the restriction of the INS to the points of* $\mathcal{C}(0)$ *is* M*, and* 0 *is its leftmost point.*

(i) *Prove that the probability* p *that* $\mathcal{C}(0)$ *has shape* M *is given by*

$$p = \sum_{a,b,c\geq 0} Cat_a Cat_b Cat_{a+b} Cat_c 4^{-2a-2b-c-4}.$$

We want to prove that this quantity is a polynomial in $\frac{1}{\pi}$ *with rational coefficients.*

(ii) *Show that it is enough to prove that*

$$p' := \sum_{a,b\geq 0} Cat_a cat_b Cat_{a+b} 4^{-2a-2b} \in \mathbb{Q}\left[\frac{1}{\pi}\right].$$

(iii) *Prove that we have*

$$p' = \sum_{n\geq 0} Cat_n Cat_{n+1} 16^{-n}.$$

(iv) Prove that

$$p' = 8\left(1 -_2 F_1\left(-\frac{1}{2}, \frac{1}{2}; 2; 1\right)\right),$$

and conclude.

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